SOLVING FOR THE NUMBER OF CASH FLOWS AND PERIODS IN FINANCIAL PROBLEMS

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***Note: Figures may be missing from this format of the document

The time value of money (NM) concepts often receive more attention in introductory finance courses than do other topics. Textbook solutions to time value of money problems usually instruct students to solve for the number of periods to the closest year However, the NM tables In textbooks show solutions for whole periods only. For this reason, these tables are often inadequate for solving some practical problems.

Alternatively, instructions are sometimes given for finding the exact number of periods using a financial calculator (Brigham [3]). Many students do not own a financial calculator, and those students using one may not gain the understanding by following an algorithm. Unlike keystrokes in a financial calculator, the use of equations enhances understanding NM.

Spreadsheet programs such as Lotus 1-2-3 have functions to solve for the number of cash flows for a beginning and ending amount (@TERM *in* Lotus) and for a stream of payments assuming end of period payments (@CTERM in Lotus), but no functions are available for solving for the number of periods of an annuity due, for the number of periods of an annuity with a balloon payment, or for the number of periods of a graduated payment annuity Equations are particularly useful for these spreadsheet applications.

This paper presents equations for determining the number of periods (n) associated with simple and annuity cash flows. Specifically, the NM algebraic equations are presented tar single payment, beginning and end of period annuities, a graduated payment annuity, and an annuity with a balloon payment. The reader will see that solving equations for the number of periods requires, at most, a calculator with a natural log transformation routine.

SINGLE CASH FLOW
The future value (FV) of a present value amount (PV) invested at k percent per period for n periods is calculated as follows:

$$FV = PV(1 + k)^n$$

The number of periods to accumulate a future value may be found by dividing FV by PV and transforming the above equation into natural logarithms as follows:

$$n = \frac{\ln(FV/PV)}{\ln(1 + k)} \qquad (1)$$

As an example, a commonly asked question is 10 determine how many years are required for an Initial investment to double if deposited in a bank that pays k percent stated interest. Suppose the future value (FV) is $400, the present value (PV) equals $200 and the interest rate k = 7 percent, compounded annually. Finance teachers sometimes offer students the rule of 72 approximation; divide 72 by the percentage as a non-decimal (7% = 7) and estimate 10.3 years to double the present value amount. The exact number of years required to accumulate the future value would be:

$$n = \frac{\ln(\$400/\$200)}{\ln(1 + 0.07)} = \frac{0.6931}{0.0677} = 10.2448 \text{ yrs.}$$

This period corresponds to 10 years and 90 clays (0.24443*365 days/year).

ANNUITY TERMS
in the case of a series of level cash flows, the number of payments is found through solving for the number of periods (n) required for the payments (PMT) to equal the present value of an annuity (PVA), or, in the case of future value, the future value of an annuity (FVA) Both ordinary and annuity-due equations are examined.

PRESENT VALUE TERM
The present value of an annuity with end-of-period payments, evaluated at k percent for (n) periods is given by the following equation:

$$PVA = PMT \sum_{t=1}^{n} \frac{1}{(1 + k)^t} \qquad (2)$$

In the above equation, each payment is equal In annuity form, Equation 2 is frequently stated as:

$$PVA = PMT \left[\frac{1 - 1/(1+k)^n}{k} \right] \qquad (3)$$

The number of periods in Equation 3 can be found by solving for the n-term as follows:

$$n = \frac{-\ln[1 - ((k)(PVA)/PMT)]}{\ln(1 + k)} \qquad (4)$$

A straight forward application of Equation 4 Is in the context of a discounted payback problem in capital budgeting (Bhandari [21]). Suppose a project has cost of $65,000 and has expected operating cash flows of $15,000 annually for 10 years. The marginal cost of capital to finance the

project is 14 percent annually. What is the discounted payback period? The answer **is 7.1** years, determined as follows:

$$n = \frac{-\ln[1 - (0.14(\$65,000)/\$15,000)]}{\ln(1 + 0.14)} = 7.1 \text{ years}$$

This equation is particularly useful with monthly payment problems, because many term solutions with monthly payments are greater than 60 periods, the limit of most time value of money tables given in textbooks.

If annuity cash flows occur at the beginning of each period instead of at end, then the right side of Equation 3 is multiplied by $(1 + k)$ as follows:

$$PVA = PMT \left[\frac{1 - 1/(1+k)^n}{k} \right] (1+k) \qquad (5)$$

Solving for the n-term, then:

$$n = 1 - \left[\frac{\ln[(1 + k) - ((k)(PVA)/PMT)]}{\ln(1 + k)} \right] \qquad (6)$$

The discounted payback information from the previous annuity example produces a discounted payback of:

$$n = 1 - \ln[(1 + 0.14) - ((0.14)(\$65,000)/\$15,000)]$$
$$= 5.8 \text{ years}$$

FUTURE VALUE TERM
Financial decisions often involve solving for the time period required to reach a financial goal. Periodic payments into a fund receive interest on each payment as they are deposited in an interest-bearing account. The term solution becomes the number of periods required for interest-earning payments (PMT) to reach the future value of an annuity (FVA) amount.

Ordinary Annuity
The future value of an annuity formula for determining the future value of a series of payments can be written:

$$FVA = PMT \sum_{t=1}^{n} (1 + k)^t \qquad (7)$$

Equation 7 can be restated as:

$$FVA = PMT \left[\frac{(1+k)^n - 1}{k} \right] \qquad (8)$$

Equation 8 can then be rewritten as:

$$n = \frac{\ln[1 + ((k)(FVA)/PMT)]}{\ln(1 + k)} \qquad (9)$$

Suppose the owner of a company can Invest $150,000 each year at a 20 percent annual stated rate but has the opportunity to invest $150,000 per year in the capital market and earn a 10 percent stated rate. The owner wants to know how much longer it will take to accumulate $1,000,000 by investing in the capital market than by investing in the company. The solution involves solving for the number of years to accumulate $1,000,000 under both alternatives. For reinvesting in the company, the number of years is 4.6 as found below:

$$n = \frac{\ln[1 + ((0.20)($1,000,000)/($150,000)]}{\ln(1 + 0.20)}$$
$$= 4.6 \text{ years}$$

Under the capital market alternative, 5.4 years are required as shown:

$$n = \frac{\ln[1 + ((0.10)($1,000,000)/($150,000)]}{\ln(1 + 0.10)}$$
$$= 5.4 \text{ years}$$

The difference in the time to accumulate $1.000,000 is 0.80 years (5.4 years minus 4.6 years).[2]

Annuity Due
If payments were made at the beginning of the period, then the right side of Equation 8 is multiplied by $(1 + k)$, since each payment receives interest for one additional period. The number-of-years solution far an annuity due becomes:

$$n = \left[\frac{\ln[(1 + k) + ((k)(FVA)/PMT)]}{\ln(1 + k)} \right] - 1 \qquad (10)$$

Because cash flows occur sooner, the term is shorter for an annuity due than for a deferred annuity. If the data from the previous annual annuity example is used for illustration, the number of years required to attain $1,000,000 for an annuity due would be:

$$n = \left[\frac{\ln[(1 + 0.1) + ((0.1)($1M)/$150,000)]}{\ln(1 + 0.1)} \right] - 1$$
$$= 5.0 \text{ years}$$

GRADUATED CASH FLOW ANNUITY TERM
Finding the term of a graduated cash flow annuity is more complex than finding that of an annuity. A solution involves rearranging an equation where the present value of payments is assumed to grow at a constant growth rate (g) and is set equal to the present value of a graduated annuity (PVGA). Many finance textbooks which discuss graduated annuities do not have tables for solving these problems and many financial calculators and spreadsheets ignore these functions.

The graduated annuity is a special case of an annuity with cash flows which grow at a constant rate. A perpetuity is easily valued by the familiar constant growth stock valuation model. To understand the term solution, suppose a graduated cash flow annuity which increases (g) percent per period until the loan is repaid. The present value of the graduated cash flow annuity is:

$$PVGA = \sum_{t=1}^{n} \left[\frac{PMT(1 + g)^t}{(1 + k)^t} \right] = \frac{PMT(1+g)}{k-g} \text{ ; as } n \rightarrow \qquad (11)$$

The present value of an infinite stream of growing payments can be found by applying the constant growth model used for finding the present value of dividends. The present value of payments for n-periods is the difference between the present value of the perpetual stream of payments beginning in period t = 1 minus the present value of the value of the perpetual stream beginning in period n + 1. This can be written as:

$$PVGA = \frac{PMT(1 + g)}{k - g} - \frac{PMT(1 + g)^{n+1}}{(k - g)(1 + k)^n} \qquad (12)$$

Solving for the number of periods cash flows (n):

$$n = \frac{\ln[1-(((PVGA)(k - g))/((PMT)(1 + g)))]}{\ln(1 + g) - \ln(1 + k)} \qquad (13)$$

For illustration purposes, suppose an investor is considering purchasing stock that just paid a $2.00 dividend last period. The dividends are expected to grow at a 5 percent annual growth rate for the foreseeable future, and the investor requires a 10 percent return on the stock. If the investor pays $15.62 for the stock, the number of years required for the stock to offer a 10 percent return, given a $15.62 price is determined as follows:

$$n = \frac{\ln[1-(($15.62)(0.1 - .05))/(($2)(1.05))]}{\ln(1.05) - \ln(1.10)}$$
$$= 10 \text{ years}$$

Equation 13 eliminates a trial-and-error approach for determining the term for a graduated cash flow annuity. Equation 13 is also appropriate for determining the discounted payback period for a steadily growing stream of cash flows.

BALLOON PAYMENT ANNUITY TERM

A large final payment is often concurrent with the last interest payment of an annuity. The last payment at maturity (M), a balloon payment, is added to Equation 3 as follows:

$$PVA = PMT \left[\frac{1 - 1/(1+k)^n}{k} \right] + \frac{M}{(1 + k)^n} \qquad (14)$$

Solving for the n-term in Equation 14:

$$n = \frac{\ln[(PMT - ((M)(k)))/(PMT - ((PVA)(k)))]}{\ln(1 + k)} \qquad (15)$$

In the context of the discounted payback problem presented earlier, suppose a project has expected operating cash flows of \$15,000 annually for 10 years and a project cost of \$65,000. When the project ends it will also have a \$10,000 nonoperating cash flow from the recovery of working capital. The project can be terminated any year after the second year. If the marginal cost of capital to finance the project is 14 percent, the discounted payback is 6.4 years, determined as follows:

$$n = \frac{\ln[(\$15,000 -((\$10,000)(.14)))]/(\$15,000 -((\$65,000)(.14)))]}{\ln(1.14)}$$
$$= 6.4 \text{ years}$$

CONCLUSION

This paper illustrates the way logarithmic transformations permit the solution of the term for many common TVM problems. Importantly, problems involving monthly payments can be solved without the use of financial tables or a financial calculator. These equations are also useful for spreadsheet analysis, since many spreadsheets ignore these financial functions.

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ENDNOTE

2. Some financial calculators round up to the nearest whole year for this calculation. The HP-12C displays n = 6 for this problem.